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Critical duality

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Abstract We look for a general framework in which the Ekeland duality can be formulated. We propose a scheme in which the parameter sets are provided with a coupling function which induces a conjugacy. The decision spaces are not supposed to have any special structure. We examine several examples. In particular, we consider some special classes of generalized convex functions.

Keywords Conjugacy · Coupling · Duality · Dual problem · Radiant function · Shady function

1 Introduction

It is well known that a number of physical phenomena can be described by a notion of stationarity rather than by the minimization or maximization of some functional. Thus, a duality pertaining to stationarity is desirable. Such a duality exists: it is the Ekeland duality ([13,14]). In [56] we made attempts to show that this duality can encompass various duality results. On the other hand, in spite of the fact that the most classical duality schemes are devoted to minimization (or maximization) procedures, one may wonder whether they have a bearing on stationarity.

It is the purpose of the present paper to give an affirmative answer to that question. In particular, we endeavour to show that some dualities of generalized convexity can be combined with the Ekeland duality. We stress two points. First, we intend to show that the Ekeland duality scheme can be performed even when the decision space X has no special structure. Second we intend to show that the linear structures on the parameter space W and it dual space W' are not necessary. Thus we allow coupling functions between W and W' which

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are quite general. Such aims have been reached for the usual duality theories using either Lagrangians or perturbations (see [54,58] and their references).

In Sect. 1 we give a version of the Ekeland duality in a framework with no linear structure. Such an extension is made in the spirit of a general stream illustrated in [45] and in numerous works (see [54] for a recent extensive bibliography). We use nonlinear coupling functions and show that the essence of the Ekeland's scheme can be preserved, what brings the possibility to apply it to augmented Lagrangian duality, quasiconvex duality, starshaped duality, submodular duality… We also look for a structure which is as bare as possible (hence, as general as possible). In particular, while we suppose that the parameter space W is paired with a dual space W' through a coupling function, we do not assume that the decision space X is anything else than a set, even if we associate to it a related space X' with a base point $0_{X'}$. In particular, X' may be the set of all functions on X, or some set $\mathcal{F}(X)$ of functions from X to $\mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$ or $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ and $0_{X'}$ the null function on X.

Since the concept of critical point may emerge from the concepts of nonsmooth analysis, we introduce two classes of functions which are defined with the help of a subdifferential. In the second one, which is symmetric, the idea of fuzziness plays a crucial role.

When a more structured framework is available in which the decision space X is also provided with a coupling function, it is possible to compare our approach to the original one by Ekeland.

We present several examples. In particular, we devote some attention to the quadratic case and to some special cases of quasiconvex analysis. Such examples may enlarge our views of duality methods which are already rich and varied (see [1–3,5–7,9–12,15–21,24–34,44–63,65–69,71–92] and the references therein and below).

2 The Ekeland duality scheme

In several books and papers (see [37,49,53,54]) the notion of duality is described in general terms as in the following definition which is adapted to optimization problems.

Definition 1 Given two sets W, W', a *duality* D between the set $\overline{\mathbb{R}}^W$ of extended real-valued functions on W and the set $\overline{\mathbb{R}}^{W'}$ of extended real-valued functions on W' is a map $D : \overline{\mathbb{R}}^W \to \overline{\mathbb{R}}^{W'}$ such that, for any family $(f_i)_{i \in I}$ in $\overline{\mathbb{R}}^W$

$$D(\inf_{i \in I} f_i) = \sup_{i \in I} D(f_i). \tag{1}$$

Such a map is clearly *antitone*: $D(f) \geq D(g)$ whenever $f, g \in \overline{\mathbb{R}}^W$ are such that $f \leq g$. It gives rise to a reverse duality given by $D'(h) := \inf\{g \in \overline{\mathbb{R}}^{W'} : D(g) \leq h\}$ for $h \in \overline{\mathbb{R}}^W$; then $D' \circ D$ (and $D \circ D'$) is *homotone*, i.e., such that $D'(D(f)) \leq D'(D(g))$ whenever $f, g \in \overline{\mathbb{R}}^W$ are such that $f \leq g$. A duality D which is compatible with addition of constants, i.e., such that $D(f + r) = D(f) - r$ for any $f \in \overline{\mathbb{R}}^W$ and any $r \in \mathbb{R}$ considered as a constant function is a tool of pleasant use. It is called a *conjugacy* (or conjugation). It can be shown that any conjugacy can be obtained as a generalized Legendre–Fenchel transform associated with some coupling function $c : W \times W' \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$, and defined by $D(f) = f^c$, where

$$f^c(w') := -\inf_{w \in w}(f(w) - c(w, w')). \tag{2}$$

Then, the reverse duality is given by a similar formula:

$$g^c(w) := -\inf_{w' \in W'} (g(w') - c(w, w')).$$

It has been shown in several papers [4,37,36,54,58,59,61] and books ([45,67,84]) that conjugacies are versatile tools. Up to now, such tools have been used for minimization or maximization problems only. In the present paper we propose to use them for the study of critical points. The transformations we will consider will go outside the framework we have just described.

The notions of critical point and of critical value can be cast in a general abstract framework in which there is no linear structure.

Definition 2 Given two sets, X, X' and a base point $0_{X'}$ of X', a point x of X is called a *critical point* of a subset J of $X \times X' \times \mathbb{R}$ if there exists some $r \in \mathbb{R}$ (called a critical value of J) such that $(x, 0_{X'}, r) \in J$. Then the pair (x, r) is called a *critical pair* of J.

The extremization of J consists in the determination of the set ext$J$ of pairs $(x, r) \in X \times \mathbb{R}$ such that $(x, 0_{X'}, r) \in J$.

In the classical case, X is a normed vector space (n.v.s.), X' is the topological dual space X^* of X and J is the one-jet of a differentiable function $j : X_0 \to \mathbb{R}$, where X_0 is an open subset of X:

$$J := \{(x, Dj(x), j(x)) : x \in X_0\}.$$

Taking the origin of X^* as a base point, we recover the usual notion. One may also suppose as in [13] that X is a differentiable manifold and replace the derivative $Dj(x)$ of j by dj_x, the restriction to the tangent space to X at $x \in X$ of the 1-form dj.

The choice of the general framework we adopt is prompted by the concepts of subdifferential. In order to conciliate the local concepts of nonsmooth analysis and the global concepts of subdifferentials linked with dualities, we adopt a general, loose notion. In general, some more conditions are required.

Definition 3 Given two sets X, X', a base point $0_{X'}$ of X', a *subdifferential* on a class $\mathcal{F}(X)$ of functions on X is a map $\partial : \mathcal{F}(X) \times X \to \mathcal{P}(X')$ with values in the space of subsets of X' which associates to a pair $(f, x) \in \mathcal{F}(X) \times X$ a subset $\partial f(x)$ of X' which is empty if x is not in the domain $\mathrm{dom}\, f := \{x \in X : f(x) \in \mathbb{R}\}$ of f.

We refer to [41,42,49,61,85] for the study of subdifferentials associated with a general duality. When a coupling function $c : X \times X' \to \overline{\mathbb{R}}$ is available, this notion is close to a classical notion, namely the Fenchel–Moreau subdifferential: for $x \in \mathrm{dom}\, f$

$$\partial^c f(x) := \{x' \in X' : c(x, x') \in \mathbb{R},\ f(\cdot) \geq c(\cdot, x') - c(x, x') + f(x)\} \tag{3}$$

and $\partial f(x) = \varnothing$ when $x \in X \setminus \mathrm{dom}\, f$. Thus $\partial^c f(x)$ is the set of linear forms x' such that there exists some associate c-affine function $c(\cdot, x') - r$, with $r \in \mathbb{R}$ which minorizes f and takes the same value at x.

In the sequel we also use the subdifferentials of nonsmooth analysis, in particular the proximal subdifferential $\partial^P j$ of j, the Fréchet (or firm) subdifferential $\partial^F j$ of j, the Dini–Hadamard (or directional) subdifferential $\partial^D j$ of j and the Clarke–Rockafellar subdifferential $\partial^C j$ of j (See [8,67]). Their values at $x \in X \setminus \mathrm{dom}\, j$ are empty and when $x \in \mathrm{dom}\, j$

they are given respectively by

$$x^* \in \partial^P j(x) \Leftrightarrow \exists c, r \in \mathbb{P} : \forall u \in B(0,r) \quad j(x+u) \geq x^*(u) + j(x) - c\,\|u\|^2,$$
$$x^* \in \partial^F j(x) \Leftrightarrow \exists \mu \in M : \forall u \in X \quad j(x+u) \geq x^*(u) + j(x) - \mu(\|u\|)\,\|u\|,$$
$$x^* \in \partial^D j(x) \Leftrightarrow \forall u \in S_X,\ \exists \mu \in M : \forall (v,t) \in X \times \mathbb{R}_+ \quad j(x+tv) \geq x^*(tv) + j(x) - \mu(\|u-v\|+t)t,$$
$$x^* \in \partial^C j(x) \Leftrightarrow \forall u \in S_X,\ \exists \mu \in M : \forall (y,v,t) \in X^2 \times \mathbb{R}_+ \quad j(y+tv) \geq x^*(tv) + j(y) - \mu(s)t,$$

where $\mathbb{P} := (0,+\infty)$, $S_X := \{u \in X : \|x\| = 1\}$, $s := \|u-v\| + \|y-x\| + t$ and M is the set of modulus, i.e., (nondecreasing) functions $\mu : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ satisfying $\lim_{r\to 0} \mu(r) = 0$; for simplicity, in the last relation we have assumed that f is continuous at x.

Given a subdifferential ∂, one can take for J the *subjet* (or hypergraph) $J^\partial j$ of a function $j : X \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ associated with ∂ :

$$J^\partial j := \{(x, x', r) \in X \times X' \times \mathbb{R} : x' \in \partial j(x), r = j(x)\}.$$

In such a case, ext$J$ is the set of pairs (x,r) such that $0_{X'} \in \partial j(x), r = j(x)$.

However, extremization problems are not limited to the preceding two cases. In particular, one may take for J some subset of the closure of a subjet with respect to some topology (or convergence) on $X \times X' \times \mathbb{R}$. Another case of interest appears when X is a n.v.s. and J is the *hypergraph* of a multifunction $F : X \rightrightarrows \mathbb{R}$ associated with a notion of coderivative or normal cone:

$$H(F) := \{(x, x^*, r) \in X \times X^* \times \mathbb{R} : (x^*, -1) \in N(G(F), (x,r)),\ r \in F(x)\},$$

where $G(F)$ is the graph of F and $N(G(F),(x,r))$ denotes the normal cone to $G(F)$ at (x,r). The *normal cone* $N(S,s)$ at s to a subset S of a n.v.s. X can be defined in different ways. When one disposes of a subdifferential ∂ one may set $N(S,s) := \mathbb{R}_+ \partial d_S(s)$, where d_S is the distance function to S: $d_S(x) := \inf\{d(x,y) : y \in S\}$ or $N(S,s) := \partial \iota_S(s)$, where ι_S is the *indicator* function of S given by $\iota_S(x) = 0$ for $x \in S$, $+\infty$ else. Correspondingly, introducing the *coderivative* $D^*F(x,r)$ of F at $(x,r) \in G(F)$ by

$$D^*F(x,r) := \{x^* \in X^* : (x^*, -1) \in N(G(F), (x,r))\},$$

we see that $H(F)$ is the set of $(x, x^*, r) \in X \times X^* \times \mathbb{R}$ such that $x^* \in D^*F(x,r)$.

The approach of Ekeland to duality ([13,14]) can be extended to the case of an arbitrary coupling by the means of the following transformation. The set W appearing in the following definition is usually a space of parameters and W' is its topological dual space if W is a n.v.s., but other cases may be considered since we do not suppose W (nor the decision space X) has a linear structure.

Definition 4 Given arbitrary sets W, W' and a coupling function $c : W \times W' \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ between W and W', the c-Ekeland (or Legendre) map $E : W \times W' \times \mathbb{R} \to W' \times W \times \mathbb{R}$ is given by

$$E(w, w', r) := (w', w, c(w,w') - r).$$

Clearly, E is a kind of involution: denoting by E' the mapping $E' : W' \times W \times \mathbb{R} \to W \times W' \times \mathbb{R}$ given by $E'(w', w, r) := (w, w', c(w,w') - r)$, one has $E \circ E' = I$, $E' \circ E = I$, so that $E^{-1} = E'$ and E' has a similar form. In particular, when $W' = W$, one has $E' = E$, and E is a genuine involution. The transform E induces a correspondence between functions on W and functions on W'. It can also be applied to multifunctions.

Definition 5 Given a coupling c between W and W', the Ekeland transform $E(J)$ of a subset J of $W \times W' \times \mathbb{R}$ is the image of $J' := E(J)$.

When J is an hypergraph, $E(J)$ is not necessarily an hypergraph. When J is the subjet $J^{\partial} f$ associated with a function f and a subdifferential ∂, the set $E(J)$ is not necessarily the subjet of some function on W'. Thus, it is of interest to introduce notions which impose part of such a requirement.

3 Some adapted classes of functions

We first delineate a criterion ensuring that a dual function can be defined.

Definition 6 ([57]) Given a coupling c between W and W' and a subdifferential $\partial : \mathcal{F}(W) \times W \to \mathcal{P}(W')$, a function $f : W \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ is a *c-∂-Ekeland function*, in short an Ekeland function, if for any $w_1, w_2 \in W,\ w' \in W'$ satisfying $w' \in \partial f(w_1) \cap \partial f(w_2)$ one has $c(w_1, w') - f(w_1) = c(w_2, w') - f(w_2)$.

Then, the (c-∂-) *Ekeland transform* of f is the function $f^E : W' \to \mathbb{R}_\infty$ given by $f^E(w') := c(w, w') - f(w)$ with $w \in (\partial f)^{-1}(w')$ for $w' \in \partial f(W)$, $f^E(w') = +\infty$ for $w' \in W' \backslash \partial f(W)$.

Thus, the graph of f^E is the projection on $W' \times \mathbb{R}$ of $E(J^{\partial} f)$.

Example Given a coupling $c : W \times W' \to \mathbb{R}$ and a base point $0_{W'}$ such that $c(w, 0_{W'}) = 0$ for all $w \in W$, let ∂^c be the subdifferential associated to c as in (3): for a function $f : W \to \overline{\mathbb{R}}$ and $w \in \operatorname{dom} f$

$$w' \in \partial^c f(w) \Leftrightarrow f(w) \in \mathbb{R}, \qquad \forall u \in W \quad f(u) \geq f(w) + c(u, w') - c(w, w').$$

Then any function is an Ekeland function since $w' \in \partial^c f(w)$ means that $f(\cdot) - c(\cdot, w')$ attains its infimum at w on W. Then, for $w' \in \partial f(W)$, one has $f^E(w') = f^c(w')$ since for all $w \in (\partial f)^{-1}(w')$ both $c(w, w')$ and $f(w)$ are finite. □

Example Any convex function (on some n.v.s.) is an Ekeland function for the Fenchel-Moreau subdifferential and $f^E = f^c$ on $\partial^c f(W)$. □

Example Any concave function on some n.v.s. W is an Ekeland function for the Fréchet and the Hadamard subdifferentials. In fact, for any $w_1, w_2 \in W,\ w^* \in W^*$ satisfying $w^* \in \partial f(w_1) \cap \partial f(w_2)$ one has $\langle w^*, w_1 \rangle - f(w_1) = \langle w^*, w_2 \rangle - f(w_2)$ since in such a case w^* is the derivative of f at w_i ($i = 1, 2$), hence $\langle w^*, w_i \rangle - f(w_i) = \min\{\langle w^*, w \rangle - f(w) : w \in W\}$. Then $f^E = -(-f)^c$ on $\partial^c(-f)(W)$ □

Example Any linear-quadratic function on W is an Ekeland function for the usual coupling function between W and its dual space W^*. This assertion, proved in ([57]), can be generalized to linear-quadratic functions which are densely defined, i.e., to functions f given by $f(w) := \frac{1}{2}\langle w, Aw \rangle - \langle w, b \rangle + c$ for some symmetric linear map $A : W \to W' := W^*$ with dense domain $D(A)$, and for some $b \in W'$, $c \in \mathbb{R}$. Let us prove that such a function f is an Ekeland function for the directional subdifferential (that will prove that it is also an Ekeland function for the firm subdifferential). We first observe that if $w' \in \partial^D f(w)$, then one has

$w' = A(w) - b$. In fact, given $w \in D(A)$, $w' \in \partial^D f(w)$, for every $u \in D(A)$ one has $\langle w', u\rangle \le f'(w, u) \le \langle A(w), u\rangle - \langle b, u\rangle$, as easily checked. Since $D(A)$ is dense in W, we conclude that $w' = A(w) - b$.

Now let $w' \in W'$, $w_1, w_2 \in X$ for $i = 1, 2$ be such that $w' \in \partial^D f(w_i)$. One has

$$\langle w_i, w'\rangle - f(w_i) = \langle w_i, Aw_i - b\rangle - \frac{1}{2}\langle w_i, Aw_i\rangle + \langle w_i, b\rangle - c = \frac{1}{2}\langle w_i, Aw_i\rangle - c$$

and, since A is symmetric and $Aw_1 = w' + b = Aw_2$, one gets

$$\langle w_1, Aw_1\rangle - \langle w_2, Aw_2\rangle = \langle w_1, A(w_1 - w_2)\rangle + \langle w_2, A(w_1 - w_2)\rangle = 0$$

Thus, $\langle w_1, w'\rangle - f(w_1) = \langle w_2, w'\rangle - f(w_2)$ and one can write $f^E(w') = \frac{1}{2}\langle A^{-1}(w' + b), w' + b\rangle - c$, even if A is non invertible. □

The following definition stems from our wish to get a more symmetric concept and a dual function which would be defined on a set larger than $\partial f(W)$. It is also motivated by the convex case in which the domain of f^E is the image of ∂f which is not necessarily convex, while a natural extension of f^E is the Fenchel conjugate whose domain is convex and which enjoys nice properties (lower semicontinuity, local Lipschitz property on the interior of its domain...). Also, this concept takes into account fuzziness, a prominent feature of nonsmooth analysis.

Definition 7 Let W and W' be normed vector spaces paired by a coupling function $c : W \times W' \to \mathbb{R}$ and let W_0, W'_0 be open subsets of W and W', respectively. A l.s.c. function $f : W_0 \to \mathbb{R}_\infty$ is said to be a (generalized) *Legendre function* for c and a subdifferential ∂ if there exists a l.s.c. function $f^L : W'_0 \to \mathbb{R}_\infty$ such that

(a) f and f^L are Ekeland functions and $f^L \mid \partial f(W_0) = f^E \mid \partial f(W_0)$;
(b) for any $w \in \mathrm{dom} f$ there is a sequence $(w_n, w'_n, r_n)_n$ in $J^\partial f$ such that $\big(w_n, \langle w_n - w, w'_n\rangle, r_n\big) \to (w, 0, f(w))$;
(b′) for any $w' \in \mathrm{dom} f^L$ there is a sequence $(w'_n, w_n, s_n)_n$ in $J^\partial f^L$ such that $\big(w'_n, \langle w_n, w'_n - w'\rangle, s_n\big) \to (w', 0, f^L(w'))$;
(c) for $w \in W_0$, $w' \in W'_0$, the relations $w' \in \partial f(w)$ and $w \in \partial f^L(w')$ are equivalent.

Condition (b) (resp. (b′)) ensures that f (resp. f^L) is determined by its restriction to $\mathrm{dom}\partial f$ (resp. $\mathrm{dom}\partial f^L$). In fact, for any $w \in \mathrm{dom} f$ one has

$$f(w) = \liminf_{u(\in \mathrm{dom}\partial f) \to w} f(u)$$

since $f(w) \le \liminf_{u\to w} f(u)$ and (b) implies $f(w) = \lim_n f(w_n)$ for some sequence $(w_n) \to w$ in $\mathrm{dom}\partial f$. Moreover, conditions (a) and (b′) imply that f^L is determined by f.

Condition (b) can be simplified when ∂f is locally bounded on the domain of f. In that case, condition (b) is equivalent to the simpler condition

(b_0) for any $w \in \mathrm{dom} f$ there exists a sequence $(w_n)_n$ in $\mathrm{dom}\, \partial f$ such that $(w_n, f(w_n)) \to (w, f(w))$.

Example Any classical Legendre function is a (generalized) Legendre function. □

Example Any l.s.c. proper convex function is a (generalized) Legendre function, as shown by the Brønsted-Rockafellar theorem. □

Example Recall that a subset C of a n.v.s. W is said to be *evenly convex* if either $C = W$, $C = \varnothing$ or if C is the intersection of a family of open half-spaces. A subset C of W is said to be *radiant* (resp. *evenly radiant*) if it is convex (resp. evenly convex) and contains 0 or is empty. A function $f : W \to \overline{\mathbb{R}}$ is said to be *radiant* (resp. *evenly radiant*) if its sublevel sets (resp. strict sublevel sets) are radiant (resp. evenly radiant) subsets. Let W' be the topological dual of W. The conjugacies associated with the coupling functions $c^{\Delta},\ c^{\wedge}$ given by

$$c^{\Delta}(w, w') = -\iota_{[w'>1]}(w), \quad c^{\wedge}(w, w') = -\iota_{[w'\geq 1]}(w)$$

are adapted to such classes of functions. In fact, for any function f on W, its conjugates $f^{\Delta},\ f^{\wedge}$ given by

$$f^{\Delta}(w') := -\inf\{f(w) : w \in W,\ \langle w, w'\rangle > 1\}$$
$$f^{\wedge}(w') := -\inf\{f(w) : w \in W,\ \langle w, w'\rangle \geq 1\}$$

are l.s.c. radiant and evenly radiant, respectively, since for all $r \in \mathbb{R}$ one has

$$[f^{\Delta} \leq r] = \{w' \in W' : \langle w, w'\rangle \leq 1\ \forall w \in [f < -r]\} = \bigcap_{w\in[f<-r]} [\langle w, \cdot\rangle \leq 1] \tag{4}$$

$$[f^{\wedge} \leq r] = \{w' \in W' : \langle w, w'\rangle < 1\ \forall w \in [f < -r]\} = \bigcap_{w\in[f<-r]} [\langle w, \cdot\rangle < 1]. \tag{5}$$

The subdifferential $\partial^{\wedge}$ associated to $c^{\wedge}$ is given by $\partial^{\wedge} f(w) := \varnothing$ if $w \notin f^{-1}(\mathbb{R})$ and, for $w \in f^{-1}(\mathbb{R})$

$$\partial^{\wedge} f(w) := \{w' \in W' : \langle w, w'\rangle \geq 1,\ f(w) = \inf f([\langle\cdot, w'\rangle \geq 1])\}$$
$$= \{w' \in W' : \langle w, w'\rangle \geq 1,\ [f < f(w)] \subset [\langle\cdot, w'\rangle < 1]\}.$$

Thus, $w' \in \partial^{\wedge} f(w)$ if, and only if, $\langle w, w'\rangle \geq 1$ and $f^{\wedge}(w') = -f(w) \in \mathbb{R}$. When f is evenly radiant, for all $w \in f^{-1}(\mathbb{R})\backslash\{0\}$, the set $\partial^{\wedge} f(w)$ is nonempty: since $w \notin [f < f(w)]$, there exist some $w' \in W'$, $r \in \mathbb{R}$ such that $\langle u, w'\rangle < r$ for all $u \in [f < f(w)]$ and $\langle w, w'\rangle \geq r$; if $[f < f(w)]$ is empty, we can secure these conditions with $r{=}1$ and otherwise we have $0 \in [f < f(w)]$, hence $r > 0$ and then $w'/r \in \partial^{\wedge} f(w)$.

Now, any function f is an Ekeland function for this coupling and the associated subdifferential. In fact, if $w' \in \partial^{\wedge} f(w_1)$ and $w' \in \partial^{\wedge} f(w_2)$ for $w_1, w_2 \in W$ we have

$$c^{\wedge}(w_i, w') - f(w_i) = -f(w_i) = -\inf f([\langle\cdot, w'\rangle \geq 1]) = f^{\wedge}(w'), \quad i = 1, 2.$$

Moreover, if f is evenly radiant, for all $w \in f^{-1}(\mathbb{R})\backslash\{0\}$ we have $f^{\wedge\wedge}(w) := (f^{\wedge})^{\wedge}(w) = f(w)$ since the set $\partial^{\wedge} f(w)$ is nonempty and for every $w' \in \partial^{\wedge} f(w)$ we have

$$-f^{\wedge\wedge}(w) := \inf\{f^{\wedge}(z) : z \in [w \geq 1]\} \leq f^{\wedge}(w') = -f(w),$$

while the inequality $f^{\wedge\wedge} \leq f$ always holds. In such a case, we also have $w \in \partial^{\wedge} f^{\wedge}(w')$. Since $f^{\wedge}$ is evenly radiant, the roles of f and $f^{\wedge}$ are symmetric and for every $w' \in (f^{\wedge})^{-1}(\mathbb{R})$ and $w \in \partial^{\wedge} f^{\wedge}(w')$, we have $w' \in \partial^{\wedge} f(w)$. The missing property to get a Legendre function on $W\backslash\{0\}$ may be provided by the following criterion. In it, we say that f is upper regular along rays if for every $w \in W\backslash\{0\}$ one $\liminf_{t\to 1_+} f(tw) \leq f(w)$. This condition is satisfied if f is upper semicontinuous (u.s.c.) along rays, i.e., if for every $w \in W\backslash\{0\}$ one has $\limsup_{t\to 1} f(tw) \leq f(w)$. It is also satisfied if f is convex along rays, i.e., if for every $w \in W\backslash\{0\}$ the function $t \mapsto f(tw)$ is convex on $\mathbb{P} := (0, +\infty)$. □

Proposition 1 *Let $f : W \to \overline{\mathbb{R}}$ be a function which is upper regular along rays. Then $f^\wedge$ is weak* lower semicontinuous (l.s.c.). If $g : W' \to \overline{\mathbb{R}}$ is coercive and weak* l.s.c., then $g^\wedge$ is u.s.c. along rays.*

If f is such that $f(w) \to -\infty$ as $w \to 0$, then $f^\wedge$ is coercive. If $g : W' \to \overline{\mathbb{R}}$ is coercive, then $g^\wedge(w) \to -\infty$ as $w \to 0$. Thus, when f is evenly radiant, $f^\wedge$ is coercive if, and only if, $f(w) \to -\infty$ as $w \to 0$.

Thus an evenly radiant function f is u.s.c. along rays and such that $f(w) \to -\infty$ as $w \to 0$ if, and only if, $f^\wedge$ is weak l.s.c. and coercive.*

Proof Let us show that for all $r \in \mathbb{R}$, the sublevel set $[f^\wedge \leq r]$ is closed when f is upper regular along rays. Let $(w'_i)_{i \in I} \to w'$ be a weak* converging net, with $w'_i \in [f^\wedge \leq r]$ for all $i \in I$. By (4), we have to show that for all $w \in [f < -r]$ we have $\langle w, w' \rangle < 1$. Now, since f is upper regular along rays, there exists some $t > 1$ such that $f(tw) < -r$. Then, for all $i \in I$ we have $\langle tw, w'_i \rangle < 1$, hence $\langle tw, w' \rangle \leq 1$ and $\langle w, w' \rangle < 1$.

Now, let us suppose g is coercive and l.s.c. and let us show that $g^\wedge$ is u.s.c. along rays. Otherwise, there exist some $w \in W \backslash \{0\}$, $\alpha > 0$ and a sequence $(t_n) \to 1$ such that $g^\wedge(t_n w) > g^\wedge(w) + \alpha$ for all $n \in \mathbb{N}$. Then $\inf g([\langle t_n w, \cdot \rangle \geq 1]) < -g^\wedge(w) - \alpha$ and we can pick some $w'_n \in [\langle t_n w, \cdot \rangle \geq 1]$ satisfying $g(w'_n) < -g^\wedge(w) - \alpha$. Since g is coercive, (w'_n) is bounded. As the closed balls of W' are weak* compact, (w'_n) has a weak* cluster point $\overline{w}'$. By weak* lower semicontinuity of g we get $g(\overline{w}') \leq -g^\wedge(w) - \alpha < -g^\wedge(w)$ and $\langle w, \overline{w}' \rangle \geq 1$, a contradiction with the definition of $g^\wedge$.

Suppose $\lim_{w \to 0} f(w) = -\infty$. Then, for all $r \in \mathbb{R}$ we can find $s > 0$ such that $f(w) < -r$ for all $w \in B(0, s)$, so that, by (5), for all $w' \in [f^\wedge \leq r]$ we have

$$\left\| w' \right\| = s^{-1} \sup\{\langle w, w' \rangle : w \in B(0, s)\} \leq s^{-1} \sup\{\langle w, w' \rangle : w \in [f < -r]\} \leq s^{-1}.$$

Thus the sublevel set $[f^\wedge \leq r]$ is bounded and $f^\wedge$ is coercive.

Conversely, suppose $g : W' \to \overline{\mathbb{R}}$ is coercive. Given $r \in \mathbb{R}$ we can find $s > 0$ such that $[g < -r] \subset B(0, s)$. Then, for all $w \in B(0, s^{-1})$ and all $w' \in [g < -r]$ we have $\langle w, w' \rangle < 1$, hence $w \in [g^\wedge \leq r]$. Thus $g^\wedge(w) \to -\infty$ as $w \to 0$. □

Corollary 1 *Every l.s.c., radiant function f on W which is upper regular along rays is a Legendre function on $W \backslash \{0\}$ for $c^\wedge$ and $\partial^\wedge$.*

Proof Since f is radiant and l.s.c., it is evenly radiant. The conjugate of such a function is also evenly radiant and l.s.c.. The other conditions of Definition 2 have been checked above. □

Example Now let us consider the case of nicely radiant functions, i.e., functions whose sublevel sets are *nicely radiant* in the sense that these sets are closed, radiant and 0 belongs to their interiors unless they are empty or reduced to $\{0\}$. Such a function attains its minimum at 0 and for all $w \in W \backslash \{0\}$ the sublevel set $[f \leq f(w)]$ contains 0 in its interior. The conjugate f^Δ of a such a function is also radiant and l.s.c. since

$$\begin{aligned} [f^\Delta \leq r] &= \{w' \in W' : \langle w, w' \rangle \leq 1 \; \forall w \in [f < -r]\} = \bigcap_{w \in [f < -r]} [\langle w, \cdot \rangle \leq 1] \\ &= [f < -r]^\Delta. \end{aligned}$$

Moreover, if f is coercive, then f^Δ is nicely radiant: for every $r \in \mathbb{R}$, there exists $s > 0$ such that $[f < -r]$ is contained in the closed ball sB_W with center 0 and radius s, so that $[f^\Delta \leq r] = [f < -r]^\Delta$ contains the ball $s^{-1} B_{W'}$. Since the subdifferential associated with

c^Δ does not seem to be adapted, let us consider the subdifferential ∂^Δ given by $\partial^\Delta f(0) = W'$, and, for $w \in W\backslash\{0\}$,

$$\partial^\Delta f(w) := \{w' \in W' : \langle w, w'\rangle = 1,\ \inf f([w' > 1]) = f(w)\}.$$

Then, every function is an Ekeland function for ∂^Δ and the usual coupling function $\langle\cdot,\cdot\rangle$ since whenever $w \in (\partial^\Delta f)^{-1}(w')$ we have $\langle w, w'\rangle - f(w) = 1 - \inf f([w' > 1])$. Thus $f^E = f^\Delta + 1$. Moreover, since $f^{\Delta\Delta} := (f^\Delta)^\Delta = f$ when f is nicely radiant, for such a function we have

$$\begin{aligned} w' \in \partial^\Delta f(w) &\Longleftrightarrow \langle w, w'\rangle = 1,\ -f^\Delta(w') = f(w) \Longleftrightarrow w \in \partial^\Delta f^\Delta(w') \\ &\Longleftrightarrow w \in \partial^\Delta f^E(w'). \end{aligned}$$

Let us show that, when f is u.s.c. along rays, condition (b) of Definition 7 is satisfied and in fact, that $\partial^\Delta f(w)$ is nonempty for all $w \in W\backslash\{0\}$. For that purpose, given $w \in W\backslash\{0\}$, we apply the Hahn–Banach theorem to separate $\{w\}$ from int$S$, where $S := [f \le f(w)]$: there exists $\overline{w}' \in W'$ such that $\langle u - w, \overline{w}'\rangle \le 0$ for all $u \in S$ and $\langle u - w, \overline{w}'\rangle < 0$ for all $u \in \operatorname{int} S$. In particular, for $u = 0$, we get $t := \langle w, \overline{w}'\rangle > 0$, so that, for $w' := t^{-1}\overline{w}'$ we have $\langle w, w'\rangle = 1$ and, since f is u.s.c. along rays $\inf f([w' > 1]) \le \inf\{f(rw) : r > 1\} \le f(w)$. On the other hand, since $\langle u, w'\rangle \le \langle w, w'\rangle = 1$ for all $u \in [f \le f(w)]$, given $v \in [w' > 1]$, we have $f(v) > f(w)$, hence $\inf f([w' > 1]) = f(w)$. □

Proposition 2 *Suppose W is a reflexive Banach space and W' is the dual of W. Then, every nicely radiant function which is u.s.c. along rays and coercive is a Legendre function for ∂^Δ and the usual coupling function $\langle\cdot,\cdot\rangle$.*

Proof As in the proof of the preceding proposition, we can show that if f is weakly l.s.c. and coercive, then f^Δ is u.s.c. along rays when W is reflexive. Thus the roles of f and f^Δ are entirely symmetric and what precedes shows that the conditions of Definition 7 are satisfied. □

4 The critical duality scheme

In this section the decision space X has no structure but with it is associated a pointed space X' with base point $0_{X'}$; the parameter space W has a base point 0_W and is coupled with some space W' by some coupling function c. The following definition introduced in [56] is reminiscent of the notion of perturbation which is one of the two main approaches to duality in convex analysis, the other one being the Lagrangian approach.

Definition 8 A subset P of $W \times X \times W' \times X' \times \mathbb{R}$ is said to be an *hyper-perturbation* of J if

$$(x, x', r) \in J \Leftrightarrow \exists w' \in W',\ (0_W, x, w', x', r) \in P\}.$$

A subset P of $W \times X \times W' \times X' \times \mathbb{R}$ is said to be a *critical hyper-perturbation* of J if

$$(x, 0_{X'}, r) \in J \Leftrightarrow \exists w' \in W',\ (0_W, x, w', 0_{X'}, r) \in P.$$

This definition is motivated by the case J is the one-jet of some function j. In such a case, if $q : W \times X \to \overline{\mathbb{R}}$ is a perturbation of j (also called a dualizing parametrization of j), i.e., a function q such that $q(0_W, x) = j(x)$ for all $x \in X$ (see [18,54,58,67]) and if q is smooth, one can take for P the one-jet of q. Then P is an hyper-perturbation of J. However, q is seldom smooth. Therefore, one is led to detect a less stringent concept.

Here, observing that in the definition of a critical hyper-perturbation the role of X' is limited to the use of its base point $0_{X'}$, we restrict our attention to a subset of the simpler product space $W \times X \times W' \times \mathbb{R}$ in which X' does not appear.

Definition 9 Given two pairs (W, W'), (X, X') of sets, base points $0_W, 0_{X'}$ of W and X', respectively, and a subset $J \subset X \times X' \times \mathbb{R}$, a subset Q of $W \times X \times W' \times \mathbb{R}$ is said to be a *critical perturbation* of J if

$$(x, 0_{X'}, r) \in J \iff \exists w' \in W',\ (0_W, x, w', r) \in Q.$$

It is said to be a *critical hemi-perturbation* of J if for all $(x, r) \in X \times \mathbb{R}$,

$$\exists w' \in W',\ (0_W, x, w', r) \in Q \Longrightarrow (x, 0_{X'}, r) \in J.$$

Note that for these two concepts J is not determined by Q; only ext$J$ is determined by Q. Also, several critical perturbations can be associated to J. If Q_1 and Q_2 are two subsets of $W \times X \times W' \times \mathbb{R}$ such that $Q_1 \cap \{0_W\} \times X \times W' \times \mathbb{R} = Q_2 \cap \{0_W\} \times X \times W' \times \mathbb{R}$, Q_2 is a critical perturbation of J whenever Q_1 is a critical perturbation of J. Given a subdifferential ∂ and a function $q : W \times X \to \overline{\mathbb{R}}$ such that $q(0_W, \cdot) = j(\cdot)$, two possible choices for a critical hemi-perturbation of J are

$$Q_0 := \{(w, x, w', q(w, x)) : 0_{X'} \in \partial q_w(x),\ w' \in \partial q_x(w)\},$$
$$Q_1 := \{(w, x, w', q(w, x)) : (w', 0_{X'}) \in \partial q(w, x)\}$$

where $q_x := q(\cdot, x)$ and $q_w := q(w, \cdot)$.

Lemma 1 *Let W and X be n.v.s. and let $q : W \times X \to \overline{\mathbb{R}}$ and $j : X \to \overline{\mathbb{R}}$ be such that $j(\cdot) = q(0_W, \cdot)$ and J is the subjet $J^\partial j$ of j.*

(a) For any subdifferential ∂, the set Q_0 is a critical hemi-perturbation of J.
(b) If ∂ is the Fréchet, the Dini–Hadamard or the Clarke–Rockafellar subdifferential, then Q_1 is a critical hemi-perturbation J.

Proof Assertion (a) is obvious. Assertion (b) is a consequence of easy chain rules. The cases of the Dini–Hadamard and the Fréchet subdifferentials are proved in [56]; let us check the case of the Clarke–Rockafellar subdifferential. Let $(w', x') \in \partial^C q(0_W, x)$. Then, for any u in the unit sphere S_X of X, we have $(0_W, u) \in S_{W \times X}$ for any of the usual norms on $W \times X$, so that there exists some modulus μ such that for every $(y, v, t) \in X^2 \times \mathbb{R}_+$ we have

$$q(0_W, y + tv) \geq x'(tv) + q(0_W, y) - \mu(\|y - x\| + \|v - u\| + t)t.$$

That shows that $x' \in \partial^C j(x)$. □

In order to study the extremization problem

$$(\mathcal{P}) \qquad \text{find } (x, r) \in X \times \mathbb{R} \text{ such that } (x, 0_{X'}, r) \in J,$$

given a coupling $c : W \times W' \to \mathbb{R}$ and a critical hemi-perturbation $Q \subset W \times X \times W' \times \mathbb{R}$ of J, considering Q as a parametrized family $(Q_x)_{x \in X}$ of subsets of $W \times W' \times \mathbb{R}$, with

$$Q_x := \{(w, w', r) : (w, x, w', r) \in Q\},$$

we can rewrite the slice of the family $(X' \times E(Q_x))_{x \in X}$ associated with 0_W (where $(E(Q_x))_{x \in X}$ is the family of Ekeland transforms of the sets Q_x's), in the form

$$Q' := \{(x', w', x, r') : x \in X,\ x' \in X',\ r\prime \in \mathbb{R},\ (w', 0_W, r') \in E(Q_x)\}.$$

Now, we consider W' as a decision space, X' as a parameter space with base point $0_{X'}$ and W as a dual space to W' with base point 0_W. This dual viewpoint leads us to introduce the set

$$J' := \{(w', w, r') \in W' \times W \times \mathbb{R} : \exists x \in X,\ (0_{X'}, w', x, r') \in Q'\}.$$

Then we get the problem

$$(\mathcal{P}') \qquad \text{find } (w', r') \in W' \times \mathbb{R} \text{ such that } (w', 0_W, r') \in J'$$

called the *adjoint problem*. The problem

$$(\mathcal{P}^*) \qquad \text{find } (w', r) \in W' \times \mathbb{R} \text{ such that } (w', 0_W, -r) \in J'$$

can be called the *dual problem* of $(\mathcal{P})$. Denoting by $\mathrm{ext}J$ the solution set of $(\mathcal{P})$, i.e., the set of $(x, r) \in X \times \mathbb{R}$ such that $(x, 0_{X'}, r) \in J$ and by $\mathrm{ext}\, J'$ the solution set of $(\mathcal{P}')$, we have the next results which use the following notation:

$$Q(w, x, r) := \{w' \in W' : (w, x, w', r) \in Q\},$$
$$Q'(x', w', r') := \{x \in X : (x', w', x, r') \in Q'\}.$$

Such a notation is convenient, but not exactly consistent with the definition of a multimap through its graph; note that when the graph takes place in a product of several factors as it is the case here, several choices are possible and here we have interchanged the variables.

Our first result shows that the dual problem can help solving the primal problem.

Proposition 3 *Let $c : W \times W' \to \mathbb{R}$ be a coupling such that $c(0_W, w') = 0$ for every $w' \in W'$ and let J be a subset of $X \times X' \times \mathbb{R}$ as above. For any critical hemi-perturbation Q of J, the set Q' is a critical perturbation of J'. Moreover, if $(w', r') \in \mathrm{ext}J'$, then $Q'(0_{X'}, w', r')$ is nonempty and for any $x \in Q'(0_{X'}, w', r')$ one has $(x, -r') \in \mathrm{ext}J$. Thus, the set of values of $(\mathcal{P}^*)$ is contained in the set of values of $(\mathcal{P})$.*

Proof The first assertion is an immediate consequence of our construction:

$$(w', 0_W, r') \in J' \Longleftrightarrow \exists x \in X :\ (0_{X'}, w', x, r') \in Q'.$$

Now, if $(w', r') \in \mathrm{ext}J'$, i.e., $(w', 0_W, r') \in J'$, there exists some $x \in X$ such that $(0_{X'}, w', x, r') \in Q'$ or $x \in Q'(0_{X'}, w', r')$ or $(w', 0_W, r') \in E(Q_x)$. Since $c(0_W, w') = 0$, that means that $(0_W, x, w', -r') \in Q$. By definition of a critical hemi-perturbation of J, that implies that $(x, 0_{X'}, -r') \in J$ or $(x, -r') \in \mathrm{ext}J$. □

A more complete result can be obtained when Q is a critical perturbation of J.

Theorem 1 *Let $c : W \times W' \to \mathbb{R}$ be a coupling such that $c(0_W, w') = 0$ for every $w' \in W'$ and let J be a subset of $X \times X' \times \mathbb{R}$ as above. For any critical hemi-perturbation Q of J, the set Q' is a critical perturbation of J'. Moreover, the problems $(\mathcal{P})$ and $(\mathcal{P}')$ are in duality in the following sense:*

(a) if $(w', r') \in \mathrm{ext}J'$, then $Q'(0_{X'}, w', r')$ is nonempty and for any $x \in Q'(0_{X'}, w', r')$ one has $(x, -r') \in \mathrm{ext}J$;
(b) if $(x, r) \in \mathrm{ext}J$, then $Q(0_W, x, r)$ is nonempty and for any $w' \in Q(0_W, x, r)$ one has $(w', -r) \in \mathrm{ext}J'$;
(c) the set of values of $(\mathcal{P})$ is the opposite of the set of values of $(\mathcal{P}')$.

Proof (a) and the first assertion have been justified above. The other assertions are immediate consequences of the definitions of Q' and J' since

$$Q' = \{(x', w', x, r') : (0_W, x, w', -r') \in Q\}$$

as $c(0_W, w') = 0$ for every $w' \in W'$. In fact, assertion (b) results from the implications

$$\begin{aligned}(x, r) \in \text{ext} J &\Leftrightarrow (x, 0_{X'}, r) \in J \\ &\Leftrightarrow \exists w' \in W' : (0_W, x, w', r) \in Q \\ &\Leftrightarrow \exists w' \in W' : (0_{X'}, w', x, -r) \in Q'\end{aligned}$$

so that for any $w' \in Q(0_W, x, r)$ one has $x \in Q'(0_{X'}, w', -r)$ i.e., $(w', -r) \in$ext J'. Assertion (c) is part of the preceding analysis. □

In spite of the apparent symmetry between assertions (a) and (b) of the preceding statement, the situation is not entirely symmetric: the parameter space X' of the adjoint problem $(\mathcal{P}')$ is not provided with a pairing with X, so that we cannot return to the primal problem $(\mathcal{P})$ by using a similar device. Such a missing link is filled in other presentations of the theory; see [13,14,56] and the short comparison of the next section.

Let us note however that symmetry can be obtained if we make a step further in stripping the Ekeland theory. Instead of assuming W and W' are paired by a coupling function and instead of considering critical perturbations, let us suppose we are given a subset R of $X \times W' \times \mathbb{R}$ such that $(x, 0_{X'}, r) \in J$ if, and only if, there exists $w' \in W'$ such that $(x, w', r) \in R$. Let us call it a *reduced perturbation* and associate to it the subset R' of $W' \times X \times \mathbb{R}$ given by

$$(w', x, r') \in R' \Longleftrightarrow (x, w', -r') \in R.$$

Then, introducing the set

$$J' := \{(w', w, r') \in W' \times W \times \mathbb{R} : \exists x \in X,\ (w', x, r') \in R'\},$$

we get an adjoint problem in considering

$$(\mathcal{P}') \qquad \text{find } (w', r') \in W' \times \mathbb{R} \text{ such that } (w', 0_W, r') \in J'.$$

Then we have a statement similar to Proposition 3; here R' is a reduced perturbation of J'.

While such a process is of utmost simplicity, it leaves open the question of constructing a reduced perturbation. The concepts of hyper-perturbation and critical perturbation, on the contrary, provide a rather natural way of getting a dual problem.

5 Comparisons with other approaches

It is on purpose that we have chosen a bare framework which assumes no special structure on the decision space X. Such a choice can be kept for the slightly different presentation in [56] which relies on the notion of critical hyper-perturbation rather than on the use of a critical perturbation. Let us clarify the relationships between these two concepts.

Rephrasing Definition 8, P is an hyper-perturbation of J if J coincides with the domain of the slice $P_0 : X \times X' \times \mathbb{R} \rightrightarrows W'$ of P given by

$$P_0(x, x', r) := \{w' \in W' : (0_W, x, w', x', r) \in P\}.$$

On the other hand, P is a critical hyper-perturbation of J if the slice $J_0 := \{(x,r) : (x, 0_{X'}, r) \in J\}$ of J coincides with the domain of the slice $P_{00} : X \times \mathbb{R} \rightrightarrows W'$ of P given by

$$P_{00}(x,r) := \{w' \in W' : (0_W, x, w', 0_{X'}, r) \in P\}.$$

Clearly, if P is an hyper-perturbation of J, then it is a critical hyper-perturbation of J.

For what concerns the relationships with critical hemi-perturbations, we have the following obvious result.

Lemma 2 *If P is a critical hyper-perturbation of J, then its slice*

$$Q := \{(w, x, w', r) \in W \times X \times W' \times \mathbb{R} : (w, x, w', 0_{X'}, r) \in P\} \tag{6}$$

is a critical perturbation of J. Conversely, for every critical perturbation Q of J, the set

$$P := \{(w, x, w', 0_{X'}, r) : (w, x, w', r) \in Q\} \tag{7}$$

is a critical hyper-perturbation of J.

The set P just described is the smallest critical hyper-perturbation of J whose associated slice is Q. The largest one is

$$\widehat{P} := \{(w, x, w', x', r) : x' \in X',\ (w, x, w', r) \in Q\}.$$

For every subset S between P and $\widehat{P}$ the associated slice is Q.

Thus the notions of critical hyper-perturbation and critical perturbation are closely related. It follows that the present duality scheme is equivalent to the one in [56]. Let us give some details about the relationships between these two presentations. In [56], for any critical hyper-perturbation P, we introduced the *partial transform* $P' := E_W(P) \subset X' \times W' \times X \times W \times \mathbb{R}$ of P given by

$$P' := \{(x', w', x, w, c(w, w') - r) : (w, x, w', x', r) \in P\}.$$

and the domain

$$J' = \{(w', w, r') \in W' \times W \times \mathbb{R} : \exists x \in X,\ (0_{X'}, w', x, w, r') \in P'\}$$

of the slice $P'_0 : W' \times W \times \mathbb{R} \rightrightarrows X$ of P' given by

$$P'_0(w', w, r') := \{x \in X : (0_{X'}, w', x, w, r') \in P'\}$$

Then, for any critical hyper-perturbation P of J, the set P' is an hyper-perturbation of J', hence is a critical hyper-perturbation of J'. When Q is related to P by (6), we obtain the same set J' as in the preceding section, hence the same dual problem. Moreover, if P is deduced from Q by relation (7), then Q' is deduced from P' by relation (6). Then Theorem 1 corresponds to [56, Thm 5].

Theorem 1 is related to [14, Prop. 3] which deals with the enlarged dual problem

$(\mathcal{E}')$ find $(w', x, r') \in W' \times X \times \mathbb{R}$ such that $(0_{X'}, w', x, 0_W, r') \in P'$.

This dual clearly corresponds to the problem

$(\mathcal{E})$ find $(x, w', r) \in X \times W' \times \mathbb{R}$ such that $(0_W, x, w', 0_{X'}, r) \in P$

via the relation $r' = -r$. [14,Prop. 3] is subsumed by the following statement. Each of its assertions is equivalent to $(0_W, x, 0_{X'}, w', -r') \in P$, hence implies that (x, r) is a solution to $(\mathcal{P})$ and (w', r') is a solution to $(\mathcal{P}')$ for $r = -r'$.

Proposition 4 *For an element (w', x, r') of $W' \times X \times \mathbb{R}$ the following assertions are equivalent:*

(a) (w', x, r') *is a solution to* $(\mathcal{E}')$;
(b) $(x, -r')$ *is a solution to* $(\mathcal{P})$ *and* $w' \in P_0(x, 0_{X'}, -r')$;
(c) (w', r') *is a solution to* $(\mathcal{P}')$ *and* $x \in P'_0(w', 0_W, r')$.

Even this enlarged framework does not allow a full symmetry: since there is no coupling between X and X', we cannot apply to $(\mathcal{E}')$ the same process. A more symmetric situation is described in [14, Prop. 3] and in the next section.

6 The full Ekeland duality

When both pairs (W, W') and (X, X') are provided with couplings denoted by c_W, c_X (or c if there is no risk of confusion) satisfying $c_W(0_W, \cdot) = 0$, $c_X(\cdot, 0_{X'}) = 0$, one can get a more complete picture. Then one provides the pair $(W \times X, X' \times W')$ with the coupling c given by

$$c((w, x), (x', w')) = c_W(w, w') + c_X(x, x')$$

This coupling enables one to use the full Ekeland transform $P'_E := E(P) \subset X' \times W' \times W \times X \times \mathbb{R}$ of P given by

$$P'_E := \{(x', w', w, x, c(w, w') + c(x, x') - r : (w, x, w', x', r) \in P\}.$$

We observe that since $c_X(\cdot, 0_{X'}) = 0$, the slice $P'_{E,0} : W' \times W \times \mathbb{R} \rightrightarrows X$ of P'_E given by

$$P'_{E,0}(w', w, r') := \{x \in X : (0_{X'}, w', w, x, r') \in P'_E\}$$

coincides with the slice P'_0 of P' considered in the preceding section and its domain

$$J'_E = \{(w', w, r') \in W' \times W \times \mathbb{R} : \exists x \in X,\ (0_{X'}, w', x, w, r') \in P'\}$$

coincides with the domain J' of the slice P'_0 of P'. Thus, the problem

$$(\mathcal{P}'_E) \qquad \text{find } (w', r') \in W' \times \mathbb{R} \text{ such that } (w', 0, r') \in J'_E$$

coincides with the adjoint problem $(\mathcal{P}')$. Denoting by $\operatorname{ext} J$ the solution set of $(\mathcal{P})$, i.e., the set of $(x, r) \in X \times \mathbb{R}$ such that $(x, 0, r) \in J$ and by $\operatorname{ext} J'_E$ the solution set of $(\mathcal{P}'_E)$, Ekeland's result is as follows. The proof is similar to the one of Theorem 1.

Theorem 2 *For any critical hyper-perturbation P of J, the set P'_E is an hyper-perturbation of J'_E, hence a critical hyper-perturbation of J'_E. Moreover, the problems $(\mathcal{P})$ and $(\mathcal{P}'_E)$ are in duality in the following sense*

(a) *the adjoint* $(\mathcal{P}''_E)$ *of* $(\mathcal{P}'_E)$ *is* $(\mathcal{P})$;
(b) *if* $(x, r) \in \operatorname{ext} J$, *then* $P_{E,0}(x, 0, r)$ *is nonempty and for any* $w' \in P_{E,0}(x, 0, r)$ *one has* $(w', -r) \in \operatorname{ext} J'_E$;
(b′) *if* $(w', r') \in \operatorname{ext} J'_E$, *then* $P'_{E,0}(w', 0, r')$ *is nonempty and for any* $x \in P'_{E,0}(w', 0, r')$ *one has* $(x, -r) \in \operatorname{ext} J$;
(c) *the set of values of* $(\mathcal{P})$ *is the opposite of the set of values of* $(\mathcal{P}'_E)$.

It is worth noting that when one disposes of couplings c_W and c_X as in the preceding theorem, the adjoint problems $(\mathcal{P}')$ and $(\mathcal{P}'_E)$ coincide. That would not be the case if the condition $c_X(\cdot, 0_{X'}) = 0$ were not satisfied.

7 The Legendre duality

A case of special interest arises when one has a critical hyper-perturbation P of J which is the subjet of some function $q : W \times X \to \overline{\mathbb{R}}$ such that for each $x \in X$ the function $q_x := q(\cdot, x)$ is an Ekeland function for a subdifferential ∂ and a coupling function c. That means that for any $w' \in W$, $w_1, w_2 \in W$ such that $w' \in \partial q_x(x_1) \cap \partial q_x(x_2)$ one has $c_W(w_1, w') - q_x(w_1) = c_W(w_2, w') - q_x(w_2)$. Then the projection on $X' \times W' \times \mathbb{R}$ of the slice of P' corresponding to x, x' is the graph of a function.

If one has a critical hemi-perturbation Q of J and for some function $q : W \times X \to \overline{\mathbb{R}}$ such that for each $x \in X$ the function $q_x := q(\cdot, x)$ is an Ekeland function for a subdifferential ∂ and a coupling function c, with

$$Q := \{(w, x, w', q(w, x)) : w \in W,\ x \in X,\ w' \in \partial q_x(w)\}$$

a similar advantage occurs. In such a case, for all $x \in X$ the set Q_x is the one-subjet of the function q_x. In particular, if q_x is a Legendre function for c and ∂, the set $E(Q_x)$ is contained in the one-subjet of the Legendre transform q_x^L of q_x and

$$J' \subset \{(w', w, r') : \exists x \in X\ (w', w, r') \in J^{\partial} q_x^L\}.$$

Thus, one is led to the extremization of the functions q_x^L.

8 Examples of duality schemes

Let us present various examples showing the versatility of the approaches presented above.

Example 1 (*convex duality*) Suppose W', W are locally convex topological vector spaces in duality. Then one can take for c the usual coupling $\langle \cdot, \cdot \rangle$ and one recovers the familiar convex duality schemes. □

Example 2 (*subaffine duality* [37,36,59–63]...) The following coupling function is more appropriate to the study of general quasiconvex problems. Given a locally convex topological vector space W with dual space W^*, taking $W' := W^* \times \mathbb{R}$, this coupling is given by

$$c_Q(w, (w^*, r)) := \langle w^*, w \rangle \wedge r \qquad (w^*, r) \in W',\ w \in W,$$

where $r \wedge s := \min(r, s)$ for $r, s \in \mathbb{R}$. Initiated in [61], a full characterization of the class of c_Q-convex functions has been given in [36, Prop. 4.2]: $f : W \to \overline{\mathbb{R}}$ is c_Q-convex iff f is lower semicontinuous, quasiconvex and for any $\lambda < \sup f(W)$ there exists a continuous affine function g such that $g \wedge \lambda \le f$. Taking the base point $0_{W'} := (0_{W^*}, 0_{\mathbb{R}})$, we see that the condition $c_Q(w, 0_{W'}) = 0$ for all $w \in W$ is satisfied, but not the condition $c_Q(0_W, w') = 0$ for all $w' \in W'$. □

Example 3 (*lower quasiconvex duality*) Suppose W', W are as in Example 1 and c is taken as

$$c_<(w, w') := -\langle w', -w \rangle_+ = \langle w', w \rangle \wedge 0 \qquad (w', w) \in W' \times W, \tag{8}$$

Note that $c_<(w, w') = c_Q(w, (w', 0))$. Now the condition $c_Q(0_W, w') = 0$ for all $w' \in W'$ is satisfied. See [36,49,60,61] for the connexion with Plastria's subdifferential ([64]).

Example 4 (*radiant conjugacies*) Let W be a n.v.s., let W' be its dual and let $c^{\wedge}$ and c^{Δ} be the couplings given by $c^{\wedge}(w, w') = -\iota_{[w' \geq 1]}(w)$ and $c^{\Delta}(w, w') = -\iota_{[w' > 1]}(w)$. These conjugacies do not satisfy the condition $c(\cdot, 0_W) = 0$. However, we have seen that we can use the conjugacies associated with these couplings simultaneously with the usual coupling and an adapted subdifferential. □

Example 5 (*shady conjugacies*) Let W be a n.v.s., let W' be its dual and let $c^{\vee}$ and c^{∇} be the couplings given by $c^{\vee}(w, w') = -\iota_{[w' \leq 1]}(w)$ and $c^{\nabla}(w, w') = -\iota_{[w' < 1]}(w)$. Then $c^{\vee}(0_W, \cdot) = 0$ and $c^{\nabla}(0_W, \cdot) = 0$. These conjugacies are of interest as they apply to quasi-convex functions f which are *shady* (or co-radiant) in the sense that $f(tw) \leq f(w)$ for all $w \in W,\ t \geq 1$. Utility functions in mathematical economics typically are opposites of such functions. See [48,53,86–89,91] for more on such functions. Criteria of the type of Corollary 1 can be devised for that class. □

Example 6 (*starshaped duality,* [48,50,72,81–83,93,94]) Let W be a convex cone in a n.v.s. and let W' be the set of continuous superlinear functions on W, c being the evaluation mapping given by $c(w', w) := w'(w)$. Then a nice and simple separation theorem (see [94] and the appendix of [54]) shows that the family of c-convex subsets containing 0 coincides with the family of closed *starshaped* subsets of W, i.e., closed subsets S such that $tx \in S$ for any $t \in [0, 1]$ and $x \in S$, so that a function f is c-convex, i.e., such that $f^{cc} = f$, if, and only if, f is l.s.c. and quasi-starshaped i.e., such that $f(tw) \leq f(w)$ for all $w \in W,\ t \in [0, 1]$.

A specialization of this duality occurs when $W' = \mathbb{R}^n$, and $W = \mathbb{R}^n_+$ and one considers the coupling function c_m defined on $W' \times W$ for $w = (w_1, \ldots, w_n)$, $w' = (w'_1, \ldots, w'_n)$ by

$$c_m(w', w) = -\max_{1 \leq i \leq n} w'_i w_i \tag{9}$$

□

Example 7 (*modular duality*) [22,23,35,43] Given a set S, let W be a subfamily of the set $\mathcal{P}_f(S)$ of finite subsets of S endowed with the order given by inclusion and let W' be a subset of the space of real-valued functions on S. Assume $\varnothing \in W$ and set for $w' \in W',\ w \in W$,

$$c(w', w) := \sum_{s \in w} w'(s)$$

with the convention $c(\varnothing, w') = 0$. This example is important for discrete optimization: when W is a distributive sublattice of $\mathcal{P}_f(S)$, c-affine functions correspond to *modular functions*, i.e., functions $f : \mathcal{P}_f(S) \to \mathbb{R}$ satisfying

$$f(A \cap B) + f(A \cup B) = f(A) + f(B) \qquad \forall A, B \in \mathcal{P}_f(S).$$

These functions are very simple as they are determined by their values on $\varnothing$ and the singletons:

$$f(A) = f(\varnothing) + \sum_{a \in A} f(\{a\}).$$

Any function $f : \mathcal{P}_f(S) \to \mathbb{R}$ which is *submodular*, i.e., such that

$$f(A \cap B) + f(A \cup B) \leq f(A) + f(B) \qquad \forall A, B \in \mathcal{P}_f(S)$$

can be extended to a convex function $\widehat{f} : \mathbb{R}^S_+ \to \mathbb{R}$ given by $\widehat{f}(w') = \sum_{i=1}^k \lambda_i f(A_i)$ if $w' = \sum_{i=1}^k \lambda_i 1_{A_i}$ with $\lambda_i \in \mathbb{R}_+ \backslash \{0\},\ A_k \subset A_{k-1} \subset \cdots \subset A_1$ are distinct subsets (see [35,Prop. 4.1]). The fact that one disposes of an adapted subdifferential, of a sandwich theorem ([23]) and of duality results ([22,38]) points out the analogies with convexity. Here the condition $c(0_W, \cdot) = 0$ is satisfied when one takes the base point $0_W := \varnothing$. □

Example 8 (*homotone duality*) Let $(W, \leq)$ be a preordered space and let W' be the set of homotone (i.e., nondecreasing) functions on W. Then the evaluation mapping $(w', w) \mapsto w'(w)$ is a coupling. The special case $W = \mathbb{R}^d$, W' being the set of superadditive homotone functions null at 0 is considered in [90]; for positively homogeneous homotone functions see [39,40,74–80]. The important case W is a distributive lattice and W' is formed of modular or submodular functions is also included in the present example.

Example 9 (*partial sublevel duality*) Let W be a n.v.s. and let $W' := W^* \times \mathbb{R}_-$. Let c^Σ be given by $c^\Sigma(w, (w^*, r)) = -\iota_{[w^* \geq r]}(w)$. This duality has been used in [49,84,91]. □

Example 10 (*augmented duality*) Given a coupling function $c : W \times W' \to \overline{\mathbb{R}}$ and a function $a : W \to \mathbb{R}$, let $c_a : W \times W' \to \overline{\mathbb{R}}$ and $\widehat{c}_a : W \times \mathbb{R} \times W' \to \overline{\mathbb{R}}$ be given by $c_a(w, w') := c(w, w') - a(w)$ and $\widehat{c}_a(w, r, w') := c(w, w') - ra(w) = c_{ra}(w, w')$. Then one can enlarge the class $\Gamma_c(W)$ of c-convex functions to the class $\Gamma_{c,a}(W) := \{f \in \overline{\mathbb{R}}^W : f + a \in \Gamma_c(W)\}$ or $\bigcup_{r>0} \Gamma_{c,ra}(W)$. In the classical case of Example 1, W being a Hilbert space, for $a(\cdot) = (1/2)\, \|\cdot\|^2$ we get the traditional augmented Lagrangian duality. Then any function f such that $f - a \in \Gamma_c(W)$ is a Legendre function. □

Other examples are given in [4,36,37,49,51,70–86,89] for instance.

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